Taylor Series : defines polynomial expansion in neighbourhood of some arbitrary point that approximates original function close to point.

Single - Variable

$$f(x+\Delta x) \simeq f(x) + \Delta x \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 f(x)}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x$$

$$+\frac{1}{24}(\Delta x)^4 \left.\frac{d^4 f(x)}{dx^4}\right|_x + \frac{1}{120}(\Delta x)^5 \left.\frac{d^5 f(x)}{dx^5}\right|_x + \frac{1}{720}(\Delta x)^6 \left.\frac{d^6 f(x)}{dx^6}\right|_x + O(\Delta x)^7$$

Multi - Variable

$$F(x+\Delta x, y, z) \simeq F(x,y,z) + \sum_{n=1}^{n=\infty} \frac{1}{n!}(\Delta x)^n \left.\frac{\partial^n F(x,y,z)}{\partial x^n}\right|_{x,y,z}$$

We can't usually calculate ∞ so we truncate to obtain 'good enough' approximation

↳ truncated series has error equal to truncated terms

↳ truncation error

Truncation error is lowest power of Δx ignored.

↳ e.g expansion up to Δx^3 is 4^{th} order

$\text{Error}(\Delta x) = \text{ERROR}(0) + c(\Delta x)^p$ → error $\propto \Delta x^p$

constant

p determined on order of acc

∴ ↑ order acc = ↑ convergence

↳

$$\frac{\text{ERROR}(\Delta x_2)}{\text{ERROR}(\Delta x_1)} = \left(\frac{\Delta x_2}{\Delta x_1}\right)^p$$

'ratio of errors between steps = ratio between steps ^{p}

We want to rearrange Taylor series for derivatives :

forward step

$$f(x+\Delta x) \simeq f(x) + \Delta x \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 f(x)}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots.$$

$$f(x+\Delta x) - f(x) \simeq \Delta x \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 f(x)}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots.$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \simeq \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x) \left.\frac{d^2 f(x)}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^2 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots.$$

finite difference formula

$$\Rightarrow \left.\frac{df(x)}{dx}\right|_x = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

1st order

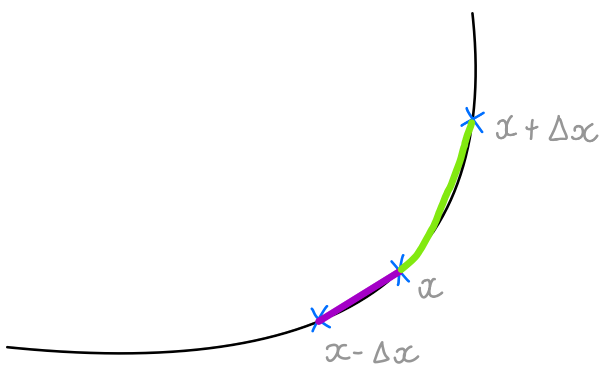
should be -ve but only || important

Can do the same but for backward step :

$$f(x-\Delta x) \simeq f(x) - \Delta x \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 f(x)}{dx^2}\right|_x - \frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots.$$

which also gives a first-order approximation

$$\left.\frac{df(x)}{dx}\right|_x = \frac{f(x) - f(x-\Delta x)}{\Delta x} + O(\Delta x)$$



Forward & Back both 1st order accurate

We can also combine the two for a central difference :

$$f(x+\Delta x) \simeq f(x) + \Delta x \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 f(x)}{dx^2}\right|_x + \frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots. \qquad (1)$$

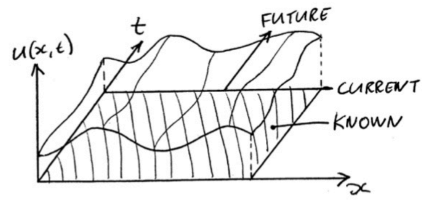
$$f(x-\Delta x) \simeq f(x) - \Delta x \left.\frac{df(x)}{dx}\right|_x + \frac{1}{2}(\Delta x)^2 \left.\frac{d^2 f(x)}{dx^2}\right|_x - \frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots. \qquad (2)$$

$$f(x+\Delta x) - f(x-\Delta x) \simeq 2\Delta x \left.\frac{df(x)}{dx}\right|_x + 2\frac{1}{6}(\Delta x)^3 \left.\frac{d^3 f(x)}{dx^3}\right|_x + \ldots. \qquad (2) - (1)$$

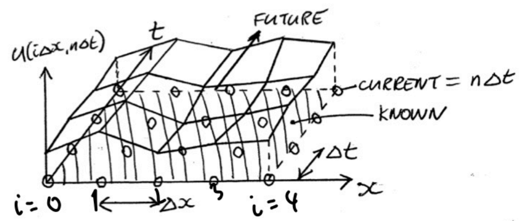
$$\frac{df(x)}{dx}\bigg|_x \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O\left((\Delta x)^2\right)$$

Central is 2nd order accurate

Discretisation of Equations :



$u(x,t)$



$$u(i\Delta x, n\Delta t) = u_i^n$$

time level

point in space

Time-marching problem : how to relate u_i^{n+1} to u_i^n for each mesh point i ?

↳ Taylor Series :

Consider expressing u_{i+1}^n in terms of quantities at constant time level n

$$u_{i+1}^n = u\left((i+1)\Delta x, n\Delta t\right) = u\left(i\Delta x + \Delta x, n\Delta t\right)$$

$\hookrightarrow \approx 0$

u_i^n

↳ Taylor → $u_{i+1}^n \simeq u(i\Delta x, n\Delta t) + \Delta x \frac{\partial u}{\partial x}\Big|_i^n + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 u}{\partial x^2}\Big|_i^n + O(\Delta x^3)$

$$\therefore \quad \frac{u_{i+1}^n - u_i^n}{\Delta x} = \frac{\partial u}{\partial x}\bigg|_i^n + \frac{1}{2}(\Delta x)\frac{\partial^2 u}{\partial x^2}\bigg|_i^n + O(\Delta x^2)$$

↳ if Δx small → $\boxed{\frac{\partial u}{\partial x}\Big|_i^n = \frac{u_{i+1}^n - u_i^n}{\Delta x}} + O(\Delta x) = \frac{u_{i+1}^n - u_i^n}{\Delta x}$

Not only choice : can have backward difference : $\boxed{\frac{\partial u}{\partial x}\Big|_i^n \approx \frac{u_i^n - u_{i-1}^n}{\Delta x}}$

Neglecting 1st order terms.

Can instead do central difference :

$$u_{i+1}^n = \cancel{u_i^n} + \Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \cancel{\frac{1}{2}(\Delta x)^2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n + \frac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + O(\Delta x^4) \quad (1)$$

$$u_{i-1}^n = \cancel{u_i^n} - \Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \cancel{\frac{1}{2}(\Delta x)^2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n - \frac{1}{6}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + O(\Delta x^4) \quad (2)$$

So (1) - (2)

$$u_{i+1}^n - u_{i-1}^n = 2\Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \frac{1}{3}(\Delta x)^3 \left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + O(\Delta x^5)$$

or

$$\left.\frac{\partial u}{\partial x}\right|_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \frac{1}{6}(\Delta x)^2 \left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + O(\Delta x^4)$$

neglect 2nd order rather than 1st, so central more accurate

Do the same for forward time difference :

$$\boxed{\left.\frac{\partial u}{\partial t}\right|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t}} + O(\Delta t)$$

→ we now can plug finite-difference derivaties into Burgers' :

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \longrightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + O(\Delta t, \Delta x^2) = 0$$

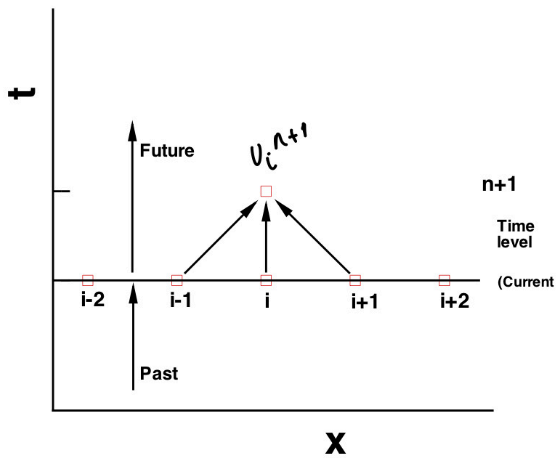
Finite-Difference Analogue

Assuming small truncation error : $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} + \dfrac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$

or $\quad u_i^{n+1} = u_i^n - \dfrac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$

FTCS - forward time, central space

(EXPLICIT)



Could alternatively use one-sided differences:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\left(u_{i+1}^n - u_i^n\right) + O(\Delta x, \Delta t) = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\left(u_i^n - u_{i-1}^n\right) + O(\Delta x, \Delta t) = 0$$

forward (FTFS)

1st order in T&S

backward (FTBS)